PROBABILITY CALCULUS FOR SILENT ELIMINATION; A METHOD FOR MEDIUM ACCESS CONTROL

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Abstract

A probability problem arising in the context of medium access control in wireless networks is considered. It is described as a problem with n urns, each one having one ball at time 0. Each ball leaves its urn after a geometrically distributed time. Then there is a first time T such that no departures take place at the times $T+1, T+2, \ldots, T+k$, where k is fixed. The focus is on the probability distribution of (X_T, S_T, T), where X_T is the number of balls that leave their urns at time T and S_T is the number of balls remaining there at that time. Efficient recursion formulas are derived. Asymptotics and continuous time approximations are considered. For $k = \infty$, T is the maximum of n geometrically distributed variables. This case has earlier got a large literature.

Keywords: Urn problem, geometric distribution, medium access control, random walk, silent period, probability generating function, recursion, exponential generating function, periodic asymptotic distribution.

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1. Introduction

The multiplicity of the maximum in a discrete random sample is of interest e.g. in the context of selection algorithms. Bruss and Grübel [4] give the following classical example: When selecting a chairperson each committee member could throw a coin repeatedly in successive rounds and leave the competition when a head is obtained. A person who last throws a head is a winner. There can be several winners.

Now, suppose that the members only announce when they throw a head? The problem might be more easily understood using the following description. There are n urns. Each urn contains one ball. At the times $1, 2, \ldots$ the ball is thrown from the urn. With probability $q = 1 - p$ it immediately returns to the urn and with probability p it leaves the urn and then nothing more happens for that urn. The balls behave independently of each other. The play stops when at k successive times no balls leave the urns; a silent k-period. Here k can be $1, 2, 3, \ldots$ An urn for which the ball is successfully thrown out from it just before the silent k-period, is a winner. To be a single winner is advantageous.

This selection algorithm is used in a medium access control for wireless networks proposed by the authors elsewhere [12]. The efficiency of such control algorithms is expressed in terms of their channel utilization, power efficiency and delay characteristics. In order to calculate these values we must answer the following probability questions:

1. What is the expected value and the probability distribution of the random time T until the first silent k-period starts? Here we may have $T = 0$ if no balls leave their urns at the times $1, 2, \ldots, k$. Otherwise $T \geq 1$ and at time T at least one ball successfully leaves its urn.

2. What is the expected value and the probability distribution of X_T, the number of urns for which the ball successfully leaves the urn at time T? If $T = 0$, then $X_T = 0$. In particular, what is the ratio $\mathbb{P}(X_T = 1)/\mathbb{P}(X_T > 0)$? This is the probability that only one urn successfully throws out the ball before the silent k-period given that at least one ball is thrown out at that time. It is the probability that an urn that 'believes' itself to be a winner is a single winner. Another measure of interest is $\mathbb{P}(X_T = 1)/(\mathbb{E}(T) + k)$, where $\mathbb{E}$ denotes expected value.

3. What is the expected value and the probability distribution of the number of urns S_T that still keep their balls at the time T described earlier? In particular, what is $\mathbb{P}(S_T = 0)$? This is the probability that the balls that leave their urns at time T are the last balls that leave.

Of course, it is also of interest to know the distribution of the random time when the last ball leaves its urn and the distribution of the number of balls which at that time leave their urns. This problem, which corresponds to the limiting case $k = \infty$ above, has successfully been treated earlier and has got a large literature; e.g. Råde [13], Eisenberg, Stengle and Strang [5], Kirschenhofer and Prodinger [10], Brands, Steutel, and Wilms [3], Baryshnikov, Eisenberg, and Stengle [2], Kirschenhofer and Prodinger [11], Bruss and Grübel [4], Jeske and Blessinger [9]. In particular the interesting asymptotics has been in focus, cf. Section 6.

In the treatment of the problem for $k = \infty$ the probability of having a single winner has got much attention. We note that it is not necessary to have a single winner since we might take additional steps to further reduce the number of winners by e.g. performing a so called backoff; cf. Wikstrand and Nilsson [14] and [6].

All the main questions above are answered in this paper but we set aside the application and its particular problems and focus on the probability distribution of the variables X_T, S_T, and T.

The paper is organized as follows. The basic model is described in Section 2. In Section 3 a general recursion formula for the probability generating function of the interesting variables is presented in a theorem. This theorem has several accompanying theorems, useful for computations. In Section 4 we look at exponential generating functions that are of help when asymptotics is considered. The latter topic is treated in Section 6. In the intermediate Section 5 several graphs are presented for illustration. The paper ends with a treatment of the continuous time case which is considerably simpler than the discrete one but often provides good approximations for that case.

2. Some preliminaries

Let S_t be the number of urns that still keep their balls just after the time t. We can see $(S_t)_0^\infty$ as a Markov process and also as a random walk on the integers, see Figure 1.

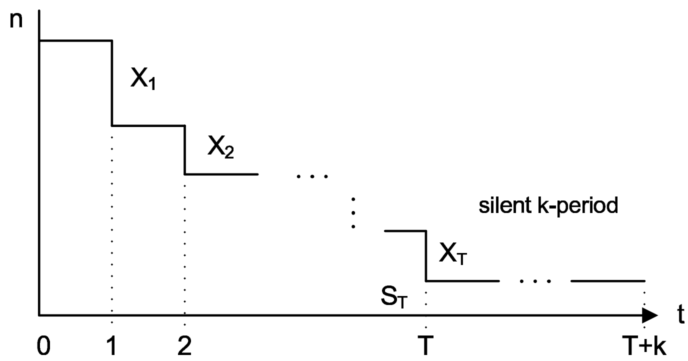

FIGURE 1: The process can be seen as a random walk on the integers.

The process starts at the point n on the vertical axis, i.e. $S_0 = n$, and then it always moves downwards so that

$$\mathbb{P}(S_t = s - x | S_{t-1} = s) = b(s, p, x), \quad \text{where} \quad b(s, p, x) = \binom{s}{x} p^x q^{s-x}, \; q = 1 - p.$$

Excluding trivial cases, we assume that $0 < p < 1$. The process stops at 0 when that point has been reached. Let $X_t = S_{t-1} - S_t$, i.e. $S_t = n - X_1 - X_2 - \cdots - X_t$. Apparently, $X_t \sim Bin(s, p)$ given $S_{t-1} = s$. The time T until the start of the silent k-period is defined as $T = \min\{t;\, X_{t+1} = X_{t+2} = \ldots X_{t+k} = 0\}$. If $T = 0$, we set $X_T = 0$ but otherwise $X_T > 0$. We notice in passing that the time $T + k$ is a stopping time since the occurrence or not of the event $T + k = t$ is determined by $S_1, S_2, \ldots, S_t$. Cf. e.g. [8, p. 492]. For $k = \infty$, T itself is such a time as $T = t$ iff $S_{t-1} > 0$ and $S_t = 0$.

Except in the case $k = \infty$, it does not seem possible to give general simple closed formulas for the desired expected values and probability distributions of X_T, S_T and T. However, it is possible to find simple and useful recursion formulas with respect to n, i.e. the starting point. To indicate the dependence on n, expected values are from now on for convenience denoted by $\mathbb{E}_n(\cdot)$ and probabilities by $\mathbb{P}_n(\cdot)$.

3. Probability generating functions

It is suitable to start with a general theorem concerning the multivariate probability generating function of (X_T, S_T, T). The argument $\mathbf{z}$ of this function is often suppressed below. The theorem can be used to get all desired results concerning the expected values and probability distributions of the variables X_T, S_T, and T.

Theorem 1. *Assume that* $S_0 = n$. *Let* $G_n = G_n(\mathbf{z}) = G_n^{X_T,S_T,T}(\mathbf{z}) = \mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T)$ *with* $|z_3| \leq 1$. *Then* $G_0 = 1$ *and*

$$G_n = q^{nk} z_2^n + z_3 \frac{1-(q^n z_3)^k}{1-q^n z_3} \left((pz_1 + q^{k+1} z_2)^n - (p + q^{k+1} z_2)^n \right)$$

$$+ z_3 \frac{1-(q^n z_3)^k}{1-q^n z_3} \sum_{x=1}^{n} b(n,p,x) G_{n-x}, \quad n = 1, 2, \ldots \tag{1}$$

Since $(1-(q^n z_3)^k)/(1-q^n z_3) = \sum_{j=0}^{k} (q^n z_3)^{j-1}$, G_n *is a polynomial in* p, q, z_1, z_2, z_3.

Proof. A successive conditioning on events $\{X_1 = X_2 = \cdots = X_{j-1} = 0, X_j = x)$ for $j = 1, 2, \ldots, k$ with $x \geq 1$ or $x = 0$, is applied. For $j = 1$ the events are interpreted as $\{X_1 = x\}$. Trivially,

$$\mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T | X_1 = X_2 = \cdots = X_k = 0) = z_1^0 z_2^n z_3^0 = z_2^n.$$

It is less obvious that, for $1 \leq j \leq k$ and $x \geq 1$,

$$\mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T | X_1 = X_2 = \cdots = X_{j-1} = 0, X_j = x) = \delta(j,x) + z_3^j \mathbb{E}_{n-x}(z_1^{X_T} z_2^{S_T} z_3^T),$$

where

$$\delta(j,x) = (z_1^x - 1) z_2^{n-x} z_3^j q^{(n-x)k} \quad \text{and} \quad \mathbb{E}_{n-x}(z_1^{X_T} z_2^{S_T} z_3^T) = G_{n-x}.$$

However,

$$\begin{aligned}
&\mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T | X_1 = X_2 = \cdots = X_{j-1} = 0, X_j = x) \\
=\ & \mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T \mathbf{1}(T = j) | X_1 = X_2 = \cdots = X_{j-1} = 0, X_j = x)) \\
+\ & \mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T \mathbf{1}(T > j) | X_1 = X_2 \cdots = X_{j-1} = 0, X_j = x).
\end{aligned}$$

The first term on the right-hand side equals $z_1^x z_2^{n-x} z_3^j q^{(n-x)k}$ since if $T = j$ the remaining $n - x$ balls must remain in their urns at k subsequent times. The second term can be rewritten as

$$\begin{aligned}
z_3^j \mathbb{E}_{n-x}(z_1^{X_T} z_2^{S_T} z_3^T \mathbf{1}(T > 0)) &= -z_3^j \mathbb{E}_{n-x}(z_1^{X_T} z_2^{S_T} z_3^T \mathbf{1}(T = 0)) + z_3^j \mathbb{E}_{n-x}(z_1^{X_T} z_2^{S_T} z_3^T) \\
&= -z_3^j z_2^{n-x} q^{(n-x)k} + z_3^j \mathbb{E}_{n-x}(z_1^{X_T} z_2^{S_T} z_3^T).
\end{aligned}$$

Collecting terms, we get the given formula.

We now turn to the main part of the proof. By the formula for total probability and since $\mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T | X_1 = x) = \delta(1,x) + z_3 G_{n-x}$ for $x > 0$, apparently

$$G_n = q^n \mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T | X_1 = 0) + \sum_{x=1}^{n} b(n,p,x)(\delta(1,x) + z_3 G_{n-x}).$$

Then the first main term can be rewritten as

$$(q^n)^2 \mathbb{E}_n(z_1^{X_T} z_2^{S_T} z_3^T | X_1 = X_2 = 0) + q^n \sum_{x=1}^{n} b(n,p,x)(\delta(2,x) + z_3^2 G_{n-x}).$$

Rewriting also the first term here and continuing in this way until we condition on the event $\{X_1 = X_2 = \cdots = X_k = 0\}$, we get

$$G_n = q^{nk} z_2^n + \sum_{x=1}^{n} b(n,p,x)\big(\sum_{j=1}^{k} \delta(j,x)(q^n)^{j-1} + \sum_{j=1}^{k} z_3 (q^n z_3)^{j-1} G_{n-x}\big).$$

Here $\sum_{j=1}^k z_3(q^n z_3)^{j-1} = z_3(1-(q^n z_3)^k)/(1-q^n z_3)$ and since $z_1^x - 1 = 0$ for $x = 0$,

$$\begin{aligned}
\sum_{x=1}^{n} b(n,p,x)\big(\sum_{j=1}^{k} \delta(j,x)(q^n)^{j-1}\big) &= \sum_{j=1}^{k}\sum_{x=1}^{n} b(n,p,x)(z_1^x - 1) z_2^{n-x} z_3^j q^{(n-x)k} (q^n)^{j-1} \\
&= \sum_{j=1}^{k} z_3^j (q^n)^{j-1} \sum_{x=0}^{n} b(n,p,x)(z_1^x - 1) z_2^{n-x} (q^k)^{n-x} \\
&= z_3 \frac{1-(q^n z_3)^k}{1-q^n z_3} \big((p z_1 + q^{k+1} z_2)^n - (p + q^{k+1} z_2)^n\big).
\end{aligned}$$

Hence the desired result in Theorem 1 follows.

Setting in Theorem 1 $z_2 = z_3 = 1$, we get a recursion formula for $G_n^{X_T}(z_1) = \mathbb{E}_n(z_1^{X_T})$. By also differentiating with respect to z_1 and setting $z_1 = 1$, we get a recursion formula for $\mathbb{E}_n(X_T)$. By differentiating y times, dividing by $y!$, and putting $z_1 = 0$, we get a recursion formula for $\mathbb{P}_n(X_T = y)$. The same procedure works for S_T and T. The results are collected below as separate theorems. The straightforward details for getting these theorems are omitted. Of course, formulas for other moments than the mean can also easily be obtained. In particular, variance formulas can be produced.

Theorem 2. *The expected values* $\mathbb{E}_n(X_T)$ *are recursively given by*

$$\mathbb{E}_n(X_T) = \frac{1-q^{nk}}{1-q^n}\big(np(p+q^{k+1})^{n-1} + \sum_{x=1}^{n} b(n,p,x)\mathbb{E}_{n-x}(X_T)\big), \quad n = 1, 2, \ldots,$$

with initial condition $\mathbb{E}_0(X_T) = 0$. *Further, for* $y > 0$,

$$\mathbb{P}_n(X_T = y) = \frac{1-q^{nk}}{1-q^n}\big(b(n,p,y)q^{(n-y)k} + \sum_{x=1}^{n} b(n,p,x)\mathbb{P}_{n-x}(X_T = y)\big),\ n = 1, 2, \ldots,$$

with initial condition $\mathbb{P}_0(X_T = y) = 0$, $y > 0$. *Moreover,* $\mathbb{P}_n(X_T = 0) = q^{nk}$.

Theorem 3. *The expected values* $\mathbb{E}_n(S_T)$ *are given by*

$$\mathbb{E}_n(S_T) = nq^{nk} + \frac{1-q^{nk}}{1-q^n}\sum_{x=1}^{n} b(n,p,x)\mathbb{E}_{n-x}(S_T), \quad n = 1, 2, \ldots,$$

with the initial condition $\mathbb{E}_0(S_T) = 0$ *for* $n = 0$. *Further*

$$\mathbb{P}_n(S_T = s) = q^{nk}\delta_{sn} + \frac{1-q^{nk}}{1-q^n}\sum_{x=1}^{n} b(n,p,x)\mathbb{P}_{n-x}(S_T = s), \quad n = 1, 2, \ldots,$$

with the initial condition $\mathbb{P}_0(S_T = s) = \delta_{s0}$, *where* δ_{sn} *is Kronecker's* δ.

Theorem 4. *We have, for* $n = 1, 2, \ldots$,

$$\mathbb{E}_n(T) = (1-q^n)\sum_{j=1}^{k} jq^{n(j-1)} \ + \ \frac{1-q^{nk}}{1-q^n}\sum_{x=1}^{n} b(n,p,x)\mathbb{E}_{n-x}(T),$$

with initial condition $\mathbb{E}_0(T) = 0$. *The first main term also equals* $\frac{1-q^{nk}}{1-q^n} - kq^{nk}$. *Moreover,*

$$\mathbb{P}_n(T = t) = q^{nk}\delta_{t0} + \sum_{x=1}^{n} b(n,p,x)\sum_{j=1}^{\min(t,k)} q^{n(j-1)}\mathbb{P}_{n-x}(T = t-j).$$

An initial condition is $\mathbb{P}_n(T = 0) = q^{nk}$ *for all* $n \geq 0$.

Remark 1. There is another recursion formula for the distribution of T. For $t \geq 1$,

$$\mathbb{P}_n(T = t) = \mathbb{P}_n(X_1 = 0, T = t) + \sum_{x=1}^{n} b(n,p,x)\mathbb{P}_{n-x}(T = t-1),$$

where $\mathbb{P}_n(X_1 = 0, T = t)$

$$= \sum_{m=0}^{\lfloor\frac{t-2}{k}\rfloor} q^{n(1+km)}\mathbb{P}_n(T = t-1-km) \ - \sum_{m=0}^{\lfloor\frac{t-k-1}{k}\rfloor} q^{nk(m+1)}\mathbb{P}_n(T = t-k(m+1)).$$

This formula can be verified by checking via Theorem 1 that the generating functions of the two sides of the equality agree. At least formally it is more efficient than that in Theorem 4 since fewer previous values are used.

Remark 2. Since the first term in the recursion formula for $\mathbb{E}_n(T)$ is close to 1 as n is large and the terms in the sum give essential contributions only for x-values around $x = np$, i.e. for $n-x$ around nq, the formula shows that $\mathbb{E}_n(T)$ asymptotically increases one unit when n is multiplied by $Q = 1/q$. Hence, $\mathbb{E}_n(T) \sim \log n / \log Q = \log_Q n$ as $n \to \infty$. One may hastily believe that X_T and S_T without normalization have asymptotic distributions as $n \to \infty$. Section 6 treats more asymptotics.

The recursion formulas (except that in Remark 1) are numerically very stable because the desired value for $n = N$ is produced by essentially a weighting with nonnegative coefficients of the previous values for $n = 0, 1, \ldots, N-1$. Of course, for n very large the calculation takes time unless also normal approximation of the binomial probabilities $b(n, p, x)$ and rough approximation of the tails of the sum are used. Stable plots using without simplification the formal complicated expressions in p and q can also be produced by a symbolic algebra system like Maple for at least all $n \leq 16$.

It is not likely that there are simple explicit formulas for the quantities considered in Theorems 2, 3, and 4 for $k < \infty$. However, for $k = \infty$, in which case T is the maximum of n geometrically distributed i.i.d. random variables, there are known such formulas, which for some completeness are collected below. They are difficult to attribute to specific authors but can be derived by simple combinatorial-algebraic methods. The formulas are not numerically stable for large n because of the sign changes.

Proposition 1. *(E.g. [11]). For $k = \infty$, we have $\mathbb{P}_n(S_T = 0) = 1$, $\mathbb{P}_n(X_T = 0) = 0$,*

$$\mathbb{P}_n(X_T = y) = \binom{n}{y} p^y \sum_{j=0}^{n-y} (-1)^j \binom{n-y}{j} \frac{1}{1-q^{j+y}}, \quad y = 1, 2, \ldots, n,$$

$$\text{and} \quad \mathbb{E}_n(X_T) = pn \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{q^j}{1-q^{j+1}}.$$

$$\text{Further,} \quad \mathbb{P}_n(T \leq t) = (1-q^t)^n \quad \text{and} \quad \mathbb{E}_n(T) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{1-q^j}.$$

4. Exponential generating functions

The recursion formulas can be transformed into functional equations using exponential generating functions (EGF); cf. e.g. Graham, Knuth, and Patashnik [7]. For a sequence $(a_n)_0^\infty$, the corresponding EGF $H_{a_n}(u)$ is given by

$$H_{a_n}(u) = \sum_{n=0}^{\infty} \frac{a_n}{n!} u^n.$$

The notation H is used in order to avoid using E too frequently. For $u > 0$, we may interpret $e^{-u} H_{a_n}(u)$ as $\mathbb{E}(a_N)$, where N is a random variable with a Poisson(u) distribution. Of course, $H_{a_n}^{(n)}(0) = a_n$. The EGF of $q^n a_n$ trivially equals $H_{a_n}(qu)$. Moreover, as is well known, for $b_n = \sum_{x=0}^{n} b(n,p,x) a_{n-x}$, we have $H_{b_n}(u) = e^{pu} H_{a_n}(qu)$. Further, to be seen later on, the asymptotic behaviour of a sequence (a_n) can be inferred from its EGF.

We now look at the EGF corresponding to $G_n = G_n(\mathbf{z})$ in Theorem 1. Multiplying the relation (1) by $1 - q^n z_3$, using that

$$\sum_{x=1}^{n} b(n,p,x) G_{n-x} = -q^n G_n + \sum_{x=0}^{n} b(n,p,x) G_{n-x},$$

and invoking the properties above, we get after some algebraic manipulations the following general theorem.

Theorem 5. *The EGF of $(G_n)_0^\infty$ satisfies the functional equation*

$$H_{G_n}(u) = \alpha(u) + \beta(u) H_{G_n}(qu) + \gamma(u) H_{G_n}(q^{k+1}u), \quad \textit{where}$$

$$\alpha(u) = \exp(q^k z_2 u) - z_3 \exp(q^{k+1} z_2 u) + z_3 \big(\exp((p z_1 + q^{k+1} z_2) u) - \exp((p + q^{k+1} z_2) u) \big)$$

$$- z_3^{k+1} \big(\exp((p q^k z_1 + q^{2k+1} z_2) u) - \exp((p q^k + q^{2k+1} z_2) u) \big),$$

$$\beta(u) = z_3 e^{pu} \quad \textit{and} \quad \gamma(u) = z_3^{k+1} (1 - \exp(p q^k u)).$$

By differentiating both sides of the functional equation in Theorem 5 with respect to z_1, z_2, or z_3 and then putting $z_1 = z_2 = z_3 = 1$ for which $H_{G_n}(u) = e^u$, we get functional equations for the EGF's of the sequences $(\mathbb{E}_n(X_T))_0^\infty$, $(\mathbb{E}_n(S_T))_0^\infty$, and $(\mathbb{E}_n(T))_0^\infty$. The equations will look as above but with $\beta(u) = e^{pu}$, $\gamma(u) = 1 - e^{pq^k u}$, and with specific $\alpha(u)$ given below:

$$\begin{aligned}
(\mathbb{E}_n(X_T))_0^\infty: \ \alpha(u) &= pu\exp((p+q^{k+1})u) - pq^k u\exp((pq^k+q^{2k+1})u), \\
(\mathbb{E}_n(S_T))_0^\infty: \ \alpha(u) &= uq^k(e^{q^k u} - qe^{q^{k+1}u}), \\
(\mathbb{E}_n(T))_0^\infty: \ \alpha(u) &= e^u + ke^{q^{k+1}u} - (k+1)e^{q^k u}.
\end{aligned}$$

The special case $k = \infty$ leads to a much simpler functional equation than that in Theorem 5 with no z_2 as $S_T = 0$:

$$H_{G_n}(u) = \alpha(u) + \beta(u)H_{G_n}(qu), \quad \text{where}$$

$$\alpha(u) = 1 - z_3 + z_3(e^{pz_1u} - e^{pu}) \quad \text{and} \quad \beta(u) = z_3e^{pu}.$$

This equation can be solved by repeated insertions on the right-hand side of new expressions for $H_{G_n}(q^ju)$, $j = 1, 2, \ldots$. We get for any integer $m \geq 1$

$$H_{G_n}(u) = \alpha(u)+b_1(u)\alpha(qu)+b_2(u)\alpha(q^2u)+\cdots+b_m(u)\,\alpha(q^mu)+b_{m+1}(u)H_{G_n}(q^{m+1}u),$$

where $b_j(u) = \prod_{i=1}^j \beta(q^{i-1}u) = z_3^j e^{(1-q^j)u}$. Since $H_{G_n}(q^{m+1}u) \to G_0(\mathbf{z}) = 1$ as $m \to \infty$, we get in case of convergence

$$H_{G_n}(u) = \sum_{j=0}^{\infty} \alpha(q^ju)z_3^j e^{(1-q^j)u} + \lim_{m\to\infty} z_3^{m+1}e^{(1-q^{m+1})u}. \tag{2}$$

Remark 3. The first sum is also obtained as limit if we start with $H_0(u) = \alpha(u)$ and then iterate according to $H_m(u) = \alpha(u)+\beta(u)H_{m-1}(qu)$, $m = 1, 2, \ldots$ Thus for $z_3 = 1$ the important last term above is lost. However, with a better start as e.g. $H_0(u) = e^u$ this simple method works to give a correct solution.

Putting $z_3 = 1$ in (2) and hence $\alpha(u) = e^{pz_1u} - e^{pu}$ for which $\alpha(q^ju)$ rapidly tends to 0 as $j \to \infty$, we get an EGF corresponding to $G_n^{X_T}$. Putting instead $z_1 = 1$ and hence $\alpha(u) = 1 - z_3$, and assuming that $|z_3| < 1$, we get an EGF corresponding to G_n^T. By differentiating these two expressions with respect to z_1 and z_3, respectively, and then putting $z_1 = 1$ and letting z_3 tend to 1, we get EGF's for $\mathbb{E}_n(X_T)$ and $\mathbb{E}_n(T)$. Below these more or less known results are collected. They can also be derived from the combinatorial fact that for $t > 0, y > 0$, $\mathbb{P}_n(X_T = y, T = t) = \binom{n}{y}(pq^{t-1})^y(1-q^{t-1})^{n-y}$.

Proposition 2. *For $k = \infty$,*

$$H_{G_n^{X_T}}(u) = e^u \sum_{j=0}^{\infty} \left(e^{-q^j(1-pz_1)u} - e^{-q^{j+1}u}\right) + e^u \quad and \quad H_{G_n^T}(u) = (1 - z_3)e^u \sum_{j=0}^{\infty} e^{-q^j u} z_3^j$$

$$H_{\mathbb{E}_n(X_T)}(u) = e^u \sum_{j=0}^{\infty} pq^j u e^{-q^{j+1}u} \quad and \quad H_{\mathbb{E}_n(T)}(u) = e^u \sum_{j=0}^{\infty} (j+1)(e^{-q^{j+1}u} - e^{-q^j u}).$$

Remark 4. For $k = \infty$, EGF's for the probabilities $\mathbb{P}_n(X_T = y)$, $n = 0, 1, 2, \ldots$, with $y \geq 1$, are then given as

$$H_{\mathbb{P}_n(X_T=y)}(u) = e^u \frac{1}{y!} \sum_{j=0}^{\infty} (pq^j u)^y e^{-q^j u}.$$

For $k < \infty$, things are more complicated but the iterative technique works to get an approximate solution in a finite number of steps and a solution as a limit. We start with e.g. $H_0(u) = e^u$ and then iterate according to

$$H_m(u) = \alpha(u) + \beta(u) H_{m-1}(qu) + \gamma(u) H_{m-1}(q^{k+1}u), \; m = 1, 2, \ldots$$

The limit equals the desired $H_{G_n}(u)$. A nice variant of this technique is to use instead the formula $H_m(u) = \alpha(u) + \beta(u)H_m(qu) + \gamma(u)H_{m-1}(q^{k+1}u)$. Equivalently, with $K_m(u) = e^{-u}H_m(u)$ and since $\beta(u) = z_3 e^{pu}$ and $\gamma(u) = z_3^{k+1}(1 - e^{pq^k u})$,

$$K_m(u) = e^{-u}\alpha(u) + z_3 K_m(qu) + z_3^{k+1}(e^{(q^{k+1}-1)u} - e^{(q^k-1)u})K_{m-1}(q^{k+1}u).$$

We start with $K_0(u) = 1$ and then solve in each step for $K_m(u)$. If q is small, already

$$K_1(u) = \sum_{i=0}^{\infty} A(q^i u) z_3^i, \quad \text{where} \;\; A(u) = e^{-u}\alpha(u) + z_3^{k+1}(e^{(q^{k+1}-1)u} - e^{(q^k-1)u}) \quad (3)$$

is an excellent approximation for $|z_3| < 1$.

From the iterations it becomes obvious that for $z_3 = 1$, $H_{G_n}(u)$, i.e. the EGF of $G_n^{X_T,S_T}$, has a representation as a limit of finite sums of functions of the form

$$\pm \exp((\lambda_0 + \lambda_1 z_1 + \lambda_2 z_2)u)$$

with $\lambda_0 \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0$, and $\lambda_0 + \lambda_1 + \lambda_2 < 1$. However, this mixture representation seems to be useful only in special cases

5. Some graphs

In this section we present some graphs which illustrate the application of the formulas in Theorems 2, 3 and 4. We let most of them speak for themselves without comments.

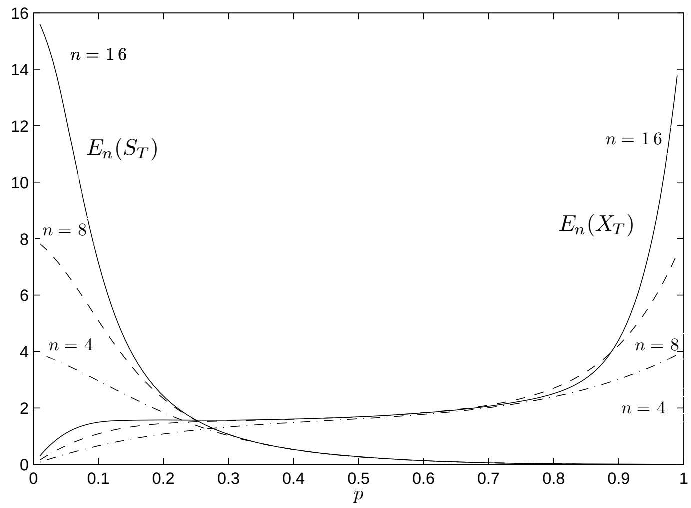

FIGURE 2: $\mathbb{E}_n(X_T)$ and $\mathbb{E}_n(S_T)$ versus p for $n = 4, 8, 16$ with $k = 2$.

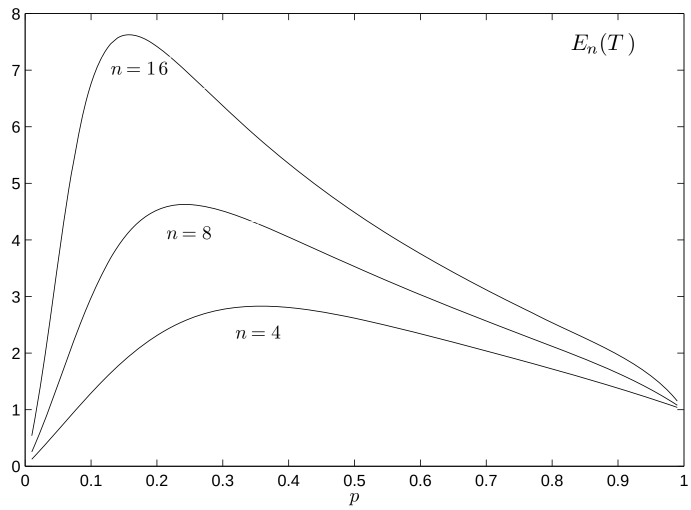

FIGURE 3: $\mathbb{E}_n(T)$ versus p for $n = 4, 8, 16$ with $k = 2$.

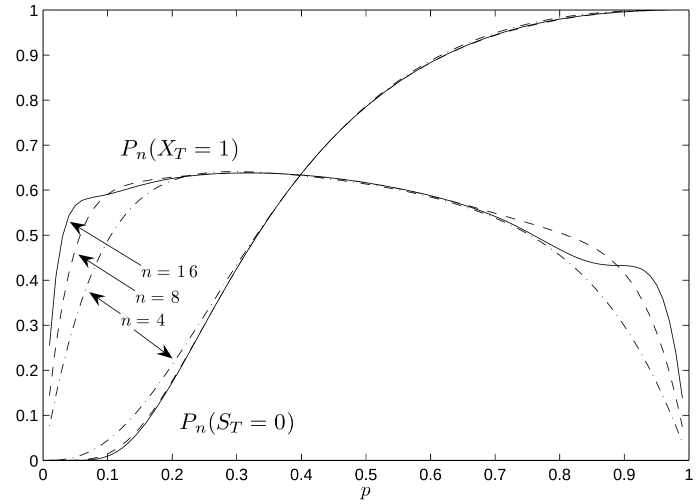

FIGURE 4: $\mathbb{P}_n(X_T = 1)$ and $\mathbb{P}_n(S_T = 0)$ versus p for $n = 4, 8, 16$ with $k = 2$.

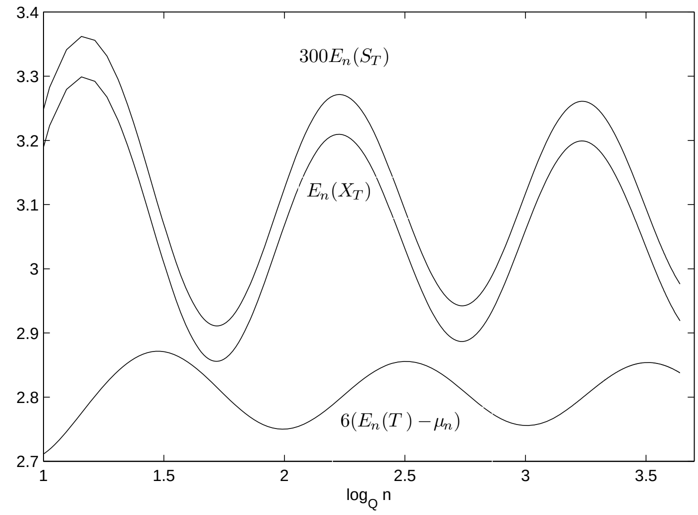

FIGURE 5: $\mathbb{E}_n(X_T)$ and normalized versions of $\mathbb{E}_n(S_T)$ and $\mathbb{E}_n(T) - \mu_n$ versus $\log_Q n$ for $q = 1/Q = 0.15$ and $k = 2$. Here $\mu_n = \sum_{j=1}^{n} j^{-1}/\log_Q n$.

We note that in Figure 4, $\mathbb{P}_n(X_T = 1)$ is very flat around $p = 1/2$ and also more or less independent of n around $p = 1/2$. Figure 4 also indicates the fact that for large n, $\mathbb{P}_n(X_T = 1)$ has a local maximum point near $p = 1$. Figure 5 shows a periodic behaviour of the quantities (versus $\log_Q n$) that will be commented more on in Section

6. We can see that the period is the same for all of them and that $\mathbb{E}_n(X_T)$ and $\mathbb{E}_n(S_T)$ have the same phase whereas $\mathbb{E}_n(T)-\mu_n$ has a phase that seems to be a quarter of a period away from that of the others.

6. Some asymptotics

As seen in Figure 5, there is a periodicity for n large. Our ambition is not to penetrate this fact in all aspects since from an application point of view it is rather unimportant unless q is close to 0. However, the periodicity is interesting as it is there and it cannot be totally neglected. As earlier mentioned, it has got a large literature for the case $k=\infty$. We start here by recalling some known results for that case.

Proposition 3. *(Bruss and Grübel, [4]). For $k=\infty$, let*

$$\pi_y(n) = \frac{(np)^y}{y!}\sum_{j\in\mathbb{Z}} q^{jy}\exp(-nq^j) \quad \text{and} \quad \tau(n) = np\sum_{j\in\mathbb{Z}} q^j \exp(-nq^{j+1}).$$

Asymptotically, $\mathbb{P}_n(X_T=y)\sim\pi_y(n),\ y\geq 1,$ and $\mathbb{E}_n(X_T)\sim\tau(n)$. The functions π_y and τ are periodic with respect to $\log_Q n$ (with $Q=1/q$) in the sense that $\pi_y(nQ)=\pi_y(n)$ and $\tau(nQ)=\tau(n)$.

Here $\pi_y(n)$ and $\tau(n)$ also work as good approximations for rather small n. The sums can be calculated by just using a rather small number of terms around $j_0=\log_Q(nq)$ for $\tau(n)$ and around $j_1=\log_Q(n/y)$ for $\pi_y(n)$. Because of the periodicity there is no asymptotic distribution for X_T.

Proposition 4. *(Kirschenhofer and Prodinger, [10, 11]). For $k=\infty$ and large n an 'exact' approximation is given by (with γ = Euler's constant, Γ = Gamma function, and $i=\sqrt{-1}$)*

$$\mathbb{P}_n(T=t)\sim\frac{p^t}{\log Q}\big(\frac{1}{t}+\delta_t(\log_Q n)\big) \ \text{with}\ \delta_t(x)=\frac{1}{t!}\sum_{j\neq 0}\Gamma(t-\frac{2\pi ij}{\log Q})e^{2\pi ijx}$$

$$\mathbb{E}_n(T)\sim\log_Q n+\frac{\gamma}{\log Q}+\frac{1}{2}+\frac{\Delta(\log_Q n)}{\log Q}, \quad \text{where} \quad \Delta(x)=-\sum_{j\neq 0}\Gamma(-\frac{2\pi ij}{\log Q})e^{2\pi ijx}.$$

The sums giving the periodic functions $\delta_t(x)$ and $\Delta(x)$, which oscillate around 0, can be calculated by using rather few terms around $j=0$. These results were derived from integral representations of the sums in Proposition 1 by skilled calculus of residues. The periodic effect is small unless $q = 1/Q$ is close to 0.

Although Bruss and Grübel [4] use other tools than EGF's to derive Proposition 3, it might be most easily understood by the help of EGF's. Let a_n be a sequence and $H(u)$ its EGF. Assuming that a_n has a limit a, trivially $H(u)/e^u \to a$ as $u \to \infty$. Then we also have $a_n \sim H(n)/e^n$ for large n. Now this result must continue to hold if the sequence $(a_n)_0^\infty$ is bounded and $a_n/a_m \to 1$ as $n, m \to \infty$ such that $n = m + O(m^{1/2})$. To see this, notice that by the central limit theorem applied to a Poisson(m) distribution with a large integer m, we have for $n_1 = m - \lfloor z_{\alpha/2}\sqrt{m}\rfloor$ and $n_2 = m + \lfloor z_{\alpha/2}\sqrt{m}\rfloor$ that

$$\sum_{n=n_1}^{n_2} \frac{m^n}{n!} e^{-m} \;\approx\; 1 - \alpha$$

whereas the corresponding two tail sums are close to $\alpha/2$. Hence, for α close to 0 and m large,

$$\frac{H(m)}{e^m} \approx \sum_{n=n_1}^{n_2} a_n \frac{m^n}{n!} \approx a_m$$

since $n = m + O(m^{1/2})$ for all $n \in [n_1, n_2]$. A more powerful result of a similar type but with additional conditions put on H can be found in Wilf [15, pp. 181–183]. Thus the results in Proposition 3 more or less follow from Proposition 2 and its Remark 4; notice that in Proposition 3 the terms in the sums are negligible for $j < 0$ and large n.

For $k < \infty$, there are no explicit expressions for the EGFs. However, in spite of that it is easy to realize that also for $k < \infty$ there is in general no asymptotic distribution. For example, consider only S_T. At least for q rather small, we then have from formula (3) that $K(u) = e^{-u} H_{G_n^{S_T}}(u)$ is closely given by

$$K(u) \approx K_1(u) = \sum_{i=0}^{\infty} A(q^i u)$$

where now

$$A(u) = e^{(q^k z_2 - 1)u} - e^{(q^{k+1} z_2 - 1)u} + e^{(q^{k+1} - 1)u} - e^{(q^k - 1)u}.$$

However, as $u \to \infty$, the function $K_1(u)$ is periodic with respect to $\log_Q u$. Hence $K(u)$ has no limit as $u \to \infty$ and therefore S_T has no asymptotic distribution. Assuming that for $|z_2| < 1$, $g_n(z_2) = G_n^{S_T}(z_2) \sim K(n) = e^{-n} H_{G_n^{S_T}}(n)$ as $n \to \infty$, we also get

$$g_n(z_2) \;\sim\; e^{(q^k z_2 - 1)n} - e^{(q^{k+1} z_2 - 1)n} + g_{\lfloor nq \rfloor}(z_2) + (e^{(q^{k+1} - 1)n} - e^{(q^k - 1)n}) g_{\lfloor q^{k+1} n \rfloor}(z_2).$$

Letting $n \to \infty$ and since $0 < q < 1$, this certainly yields $g_n(z_2) \sim g_{\lfloor qn \rfloor}(z_2)$, which shows the periodicity of the distribution with respect to $\log_Q n$ for large n.

Of course, a reasonable approximation of e.g. $\mathbb{E}_n(S_T)$ would then be:

$$\mathbb{E}_n(S_T) \approx \sum_{j;\,|j|\le M} c_j \exp(2\pi i j \log_Q n),$$

with M rather small and the c_j:s (with $c_{-j} = \bar{c}_j$) suitably calculated. Another approximation, but in the spirit of [4], would follow from a generalization of the following specific exponential representation result, which we state without proof. For $k = 1$ and $p = 1/2$,

$$H_{\mathbb{E}_n(S_T)}(u) = ue^u \sum_{j=0}^{\infty} \epsilon_j \sum_{m;\,2^m > 2j+1} \exp(-\frac{2j+1}{2^m}u)\frac{1}{2^m},$$

where the first 4 elements of the sequence $(\epsilon_j)_0^\infty$ are $1, -1, -1, 1$ and for $j \ge 1$, $\epsilon_{4j} = \epsilon_{4j+3} = \epsilon_j$, and $\epsilon_{4j+1} = \epsilon_{4j+2} = -\epsilon_j$.

Substituting $u = n$ and modifying by summing over all $m \in \mathbb{Z}$, we get that asymptotically

$$\mathbb{E}_n(S_T) \sim \frac{H_{\mathbb{E}_n(S_T)}(n)}{e^n} \sim \sum_{j=0}^{\infty} \epsilon_j P_j(n), \text{ where } P_j(n) = n \sum_{m \in \mathbb{Z}} \exp(-\frac{2j+1}{2^m}n)\frac{1}{2^m}.$$

The functions P_j are periodic in the sense that $P_j(2n) = P_j(n)$. For numerical computations the number of terms used in the $j-$sum should preferably be divisible by 4. For 32 terms a very good precision is obtained. Of course, for $p = 1/2$ the periodic effect is very small. Its amplitude is of order 10^{-5}.

7. Comparisons with the continuous time case

For some of the distributions and expected values encountered, good approximations can be derived by just replacing the geometric distribution for the time each ball stays in its urn by an exponential distribution with rate $\lambda = \log Q$, where $Q = 1/q$. These approximations should intuitively work well at least if p is small. Here we treat the continuous case with comparisons with the more complicated discrete case. Let ball i, $i = 1, 2, \ldots, n$, leave its urn at a random time Y_i that is exponentially distributed with mean $1/\lambda$. Then $\mathbb{P}(Y_i \le t) = 1 - e^{-\lambda t}$.

We first consider the time T when the last ball leaves its urn, corresponding to $k = \infty$ earlier. Since $\mathbb{P}_n(T \leq t) = (1 - e^{-\lambda t})^n$, we get

$$\mathbb{E}_n(T) = \int_0^\infty \mathbb{P}_n(T > t)dt = \int_0^\infty (1 - (1 - e^{-\lambda t})^n)dt = \frac{1}{\lambda} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j}.$$

However, by the lack of memory property of the exponential distribution, $T = \sum_{i=1}^n Z_i$, where the variables Z_i are independent and exponentially distributed and moreover $\mathbb{E}(Z_i) = (n - i + 1)/\lambda$. Hence more simply

$$\mathbb{E}_n(T) = \lambda^{-1} \sum_{j=1}^{n} \frac{1}{j}.$$

Setting $\bar{F}(t) = 1 - (1 - e^{-\lambda t})^n$ with $\lambda = \log Q$, we have $\mathbb{E}_n(T) = \int_0^\infty \bar{F}(t)dt$, whereas in the discrete case with a geometric distribution for each Y_i we have $\mathbb{E}_n(T) = \sum_{j=0}^\infty \bar{F}(j)$ since in that case $\mathbb{P}_n(T > j) = 1 - (1 - q^j)^n$. However, by the Euler-Maclaurin formula, cf. Abramowitz and Stegun [1, pp. 806], for any $m \geq 1$,

$$\sum_{j=0}^{\infty} \bar{F}(j) = \int_0^\infty \bar{F}(t)dt + \frac{1}{2}(\bar{F}(\infty) + \bar{F}(0))$$

$$+ \sum_{\nu=1}^{m-1} \frac{1}{(2\nu)!} B_{2\nu} \left(\bar{F}^{(2\nu-1)}(\infty) - \bar{F}^{(2\nu-1)}(0) \right) + \frac{1}{(2m)!} B_{2m} \sum_{j=0}^{\infty} \bar{F}^{(2m)}(j + \theta),$$

where $0 < \theta < 1$ and $B_{2\nu}$ are Bernoulli numbers. Now $\bar{F}(\infty) = 0$, $\bar{F}(0) = 1$, and $\bar{F}^{(2\nu-1)}(\infty) = 0$ for all $\nu \geq 1$. Moreover, $\bar{F}^{(2\nu-1)}(0) = 0$ at least if $2\nu - 1 < n$. This indicates that in the discrete case the approximation

$$\mathbb{E}_n(T) \approx \frac{1}{\log Q} \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{2}$$

is very good except if q is close to 0, which numerical studies confirm. The periodicity effect in Proposition 4 is small. Notice that $\sum_{j=1}^n j^{-1} = \log n + \gamma + O(n^{-1})$.

Let now T be the time when the first silent period of length $> a$ starts. Here a corresponds to k in the discrete case or rather to $k + 1/2$ if we think of the continuous case as discretized. Let also as before X_T be the number of balls that at time T leave their urns and let S_T be the number of balls remaining there at that time.

Apparently X_T equals 0 or 1. It equals 0 if and only if the silent period starts at time 0. Hence

$$\mathbb{P}_n(X_T = 1) = 1 - \mathbb{P}_n(X_T = 0) = 1 - e^{-a\lambda n}.$$

However, this distribution does not give us any useful approximation for the discrete case since it approximates the interesting probability $\mathbb{P}_n(X_T \geq 2)$ by 0.

We now turn to the distribution of S_T. It can be derived without big efforts and so can the expected value of T as will be seen. We let T_1 be the time when the first ball leaves its urn and we let T_2 be the length of the subsequent time interval until the second ball leaves its urn, etc. Let also I_1 be 1 if $T_1 < a$ and otherwise 0. Similarly we set $I_2 = 1$ if $T_2 < a$, etc. The I-variables are independent. Then, for $s = 0, 1, \ldots, n$,

$$\mathbb{P}_n(S_T = s) = \mathbb{P}_n(I_1 = I_2 = \cdots = I_{n-s} = 1, I_{n-s+1} = 0) = e^{-s\lambda a} \prod_{i=s+1}^{n} (1 - e^{-i\lambda a}).$$

We also have

$$\mathbb{P}_n(S_T \leq s) = \prod_{i=s+1}^{n} (1 - e^{-i\lambda a}), \quad s = 0, 1, \ldots, n.$$

Simple recursion formulas with respect to n or with respect to s, are immediate for these probabilities. Asymptotically, $\mathbb{P}_n(S_T = s) \to e^{-s\lambda a} \prod_{i=s+1}^{\infty}(1 - e^{-i\lambda a})$ as $n \to \infty$. Of course, expressions for $\mathbb{E}_n(S_T)$ and its limit as $n \to \infty$ are also immediate. There is also the recursion formula $\mathbb{E}_n(S_T) = ne^{-n\lambda a} + (1 - e^{-n\lambda a})\mathbb{E}_{n-1}(S_T)$.

As to the random time T, we have $T = \sum_{j=1}^{n} T_j I_j I_1 \cdots I_{j-1}$. Because of the lack of memory property of the exponential distribution,

$$\mathbb{E}_n(T) = \sum_{j=1}^{n} \mathbb{E}(T_j I_j)\mathbb{E}(I_1 I_2 \cdots I_{j-1}) = \sum_{j=1}^{n} \mathbb{E}_n(T_{n-j+1} I_{n-j+1})\mathbb{E}_n(I_1 I_2 \cdots I_{n-j})$$

$$= \sum_{j=1}^{n} \int_0^a tj\lambda e^{-j\lambda t} dt \prod_{i=j+1}^{n} (1 - e^{-i\lambda a}) = \sum_{j=1}^{n} ((\frac{1 - e^{-j\lambda a}}{j\lambda} - ae^{-j\lambda a}) \prod_{i=j+1}^{n} (1 - e^{-i\lambda a}).$$

We can rewrite $\mathbb{E}_n(T)$ as $\mathbb{E}_n(T) = \lambda^{-1} \sum_{j=1}^{n} j^{-1}(1 - \mathbb{P}_n(S_T \geq j)) - a\mathbb{P}_n(S_T > 0)$.

Putting $\lambda = \log Q$ in the expression for $\mathbb{E}_n(S_T)$ and choosing $a = k + 1/2$ (instead of $a = k$), we get for all q and all n numerical values that well agree with those given by the recursion formula in the discrete case for $\mathbb{E}_n(S_T)$. Actually $a = k + 1/3$ gives slightly better agreement. Of course, the periodicity effect cannot be obtained. With $a = k + 1/2$ a fairly good approximation of the discrete $\mathbb{E}_n(T)$ is also obtained for all $k \geq 2$ and all q if $1/2$ is added to the continuous $\mathbb{E}_n(T)$.

8. Final comments

From the medium access control point of view there are several issues not treated here. For example, how should k be chosen in an optimal way? And how should p be chosen? Answers are given elsewhere [12]. From the pure probability point of view there are also remaining questions. For example, we would like to have simple explicit asymptotic formulas for the probability distribution of (X_T, S_T, T) that can fully describe what is seen in Figure 5 including the phase shift. We may return to that problem later on. Finally, it should be mentioned that the methods used in this paper can be extended to permit calculation of the distribution of $(X_T, S_T, T, X_{T^*}, T^*)$, where here T^* denotes T for $k = \infty$.

Acknowledgement

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References

[1] Abramowitz, M. and Stegun, I. (1964). *Handbook of Mathematical Functions.* US Department of Commerce, Washington.

[2] Baryshnikov, Y., Eisenberg, B. and Stengle, G. (1995). A necessary and sufficient condition for the existence of the limiting probability of a tie for the first place. *Statist. Prob. Lett.* **23,** 203–209.

[3] Brands, J. J. A. M., Steutel, F. and Wilms, R. (1994). On the number of maxima in a discrete sample. *Statist. Prob. Lett.* **20,** 209–218.

[4] Bruss, F. and Grübel, R. (2003). On the multiplicity of the maximum in a discrete random sample. *Ann. Appl. Prob.* **13,** 1252–1263.

[5] Eisenberg, B., Stengle, G. and Strang, G. (1993). The asymptotic probability of a tie for the first place. *Ann. Appl. Prob.* **3,** 731–745.

[6] ETSI (1998). *Broadband Radio Access Networks (BRAN); High Performance Radio Local Area Network (HIPERLAN) Type 1; Functional Specification.* No. EN300 652 V1.2.1 ed. ETSI, Sophia Antipolis Cedex, France.

[7] Graham, R., Knuth, D. and Patashnik, O. (1989). *Concrete Mathematics.* Addison Wesley, Reading, Massachusetts.

[8] Gut, A. (2005). *Probability: A Graduate Course.* Springer Science + Business Media, Inc, New York.

[9] Jeske, D. R. and Blessinger, T. (2004). Tunable approximations for the mean and variance of the maximum of heterogeneous geometrically distributed random variables. *Amer. Statist.* **58,** 322–327.

[10] Kirschenhofer, P. and Prodinger, H. (1993). A result in order statistics related to probabilistic counting. *Computing* **51,** 15–27.

[11] Kirschenhofer, P. and Prodinger, H. (1996). The number of winners in a discrete geometrically distributed sample. *Ann. Appl. Prob.* **6,** 687–694.

[12] Nilsson, T., Wikstrand, G. and Bondesson, L. (2007). Silent elimination multiple access: An efficient channel bursting protocol. *Technical report.* Department of Computing Science, Umeå University, Sweden.

[13] Råde, L. (1991). Problem E3436. *Amer. Math. Monthly* **99,** 366.

[14] Wikstrand, G. and Nilsson, T. (2007). Untruncated eliminations in the EY-NPMA MAC protocol: Performance and optimality. *IEEE Comm. Lett.* **11,** 213–215.

[15] Wilf, H. (1994). *Generatingfunctionology, 2nd edn.* Academic Press, San Diego.